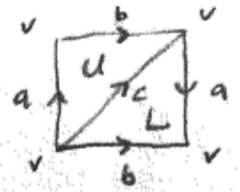
MATH 215B HW4

1. 

$\partial a = 0$

$\partial b = 0$

$\partial c = 0$

$\partial U = b - c + a$

$\partial L = a - b + c$

$H_0 \cong \mathbb{Z}$

$H_1 \cong \mathbb{Z}^{\{a,b,c\}} / \langle 2a, b-c \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2$

$H_2 = \ker \partial_2 = 0$

---

2. We get 1 vertex, $(n+1)$ edges $a_0, \dots, a_n$, $a_i = [v_0, v_1]$ in Δ^2_i, and $(n+1)$ 2-simplices $b_0, \dots, b_n$.

$\partial a_i = 0$

$\partial b_0 = a_0$

$\partial b_i = (\underset{[v_1,v_2]}{a_i} - \underset{[v_0,v_2]}{a_{i-1}} + \underset{[v_0,v_1]}{a_i}) = 2a_i - a_{i-1}$

$H_0 \cong \mathbb{Z}$

$H_1 \cong \mathbb{Z}/2^n$, generated by a_n

$H_2 \cong 0$

3.
$$\begin{tikzcd} & X \arrow[dr, "\rho"] & \\ A \arrow[ur, "i"] \arrow[rr, equal] & & A \end{tikzcd}$$

$\Big\downarrow H_n$

$$\begin{tikzcd} & H_n(X) \arrow[dr, "\rho_*"] & \\ H_n(A) \arrow[ur, "i_*"] \arrow[rr, equal] & & H_n(A) \end{tikzcd}$$

$\Rightarrow i_*$ is injective

4.
$(a, b) \longmapsto a + 2b$

$1 \longmapsto (2, 1)$

$0 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \oplus \mathbb{Z}/2 \to \mathbb{Z}/4 \to 0$

These maps define a short exact sequence, as easily checked. With careful casework and the structure theorem for fin. gen. abelian groups, we may deduce that

$0 \to \mathbb{Z}/p^m \to A \to \mathbb{Z}/p^n \to 0$

$\boxed{\Rightarrow A \cong \mathbb{Z}/p^{m+n} \text{ or } \mathbb{Z}/p^a \oplus \mathbb{Z}/p^b, \quad a \geq \max(m,n)}$

and

$0 \to \mathbb{Z} \to A \to \mathbb{Z}/n \to 0$

$\boxed{\Rightarrow A \cong \mathbb{Z} \oplus \mathbb{Z}/d \text{ where } d \mid n}$

---

5. $C = 0 \Rightarrow \ker(C \to D) = C$

$\Leftrightarrow \operatorname{im}(C \to D)$ is 0

$\Leftrightarrow D \hookrightarrow E$ is injective

and so on for the other cases. Therefore, $H_n(X,A) = 0 \;\forall n$ iff $H_n(A) \to H_n(X)$ is injective and surjective for all n, iff $A \to X$ is a homology isomorphism.

---

6. (a) A meets each path component of X iff $H_0 A \to H_0 X$ is surjective iff $H_0(X,A) = 0$.

(b) As before, but $H_0 A \to H_0 X$ is injective iff each component of X contains at most one component of A. (since $H_0 A \to H_0 X$ is

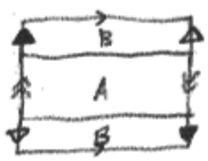
$$\bigoplus_{\substack{\text{components}\\ \text{of } A}} \mathbb{Z} \to \bigoplus_{\substack{\text{components}\\ \text{of } X}} \mathbb{Z} \Big)$$

7. Comparing universal properties, $H_1(X)$ is the pushout of

$$\begin{tikzcd} H_1(A\cap B) \arrow[r] \arrow[d] & H_1(A) \\ H_1(B) & \end{tikzcd}$$

or $H_1(X) \cong H_1(A)\oplus H_1(B) / \langle (x,-x) : x \in H_1(A\cap B)\rangle$

The Klein bottle is a union of two Möbius bands A, B



$A\cap B \simeq S^1$

so $H_1(K)$ is the pushout

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & H_1(K) \end{tikzcd}$$

or $\langle a,b \mid a^2=b^2\rangle^{ab} \cong \mathbb{Z}\oplus\mathbb{Z}/2$.

8. $$0 \to \mathbb{Z} \xrightarrow{\cong} H_2(S^2,A) \to 0 \to 0 \to H_1(S^2,A) \to \mathbb{Z}^{|A|} \twoheadrightarrow \mathbb{Z} \xrightarrow{0} H_0(S^2,A) \to 0$$

so

$H_2(S^2,A) \cong \mathbb{Z}$

$H_1(S^2,A) \cong \mathbb{Z}^{|A|-1}$

$H_0(S^2,A) \cong 0$

9. Using the LES for $SX \cong CX/X$, we get

$$\cdots \to \tilde H_n(CX) \to \tilde H_n(SX) \to \tilde H_{n-1}(X) \to \tilde H_{n-1}(CX)$$

$$0 \to \tilde H_n(SX) \xrightarrow{\cong} \tilde H_{n-1}(X) \to 0$$

Taking n cones, collapse one to get a wedge of $(n-1)$ copies of SX, so its reduced homology is

$$\tilde H_{*-1}(X)^{\oplus n-1} = \underbrace{\tilde H_{*-1}(X)\oplus \cdots \oplus \tilde H_{*-1}(X)}_{n-1 \text{ times}}$$

10. Inductively these are all true for $X^{(n-1)}$. Using the LES

$$\tilde H_{k+1}(\vee S^n) \to \tilde H_k(X^{(n-1)}) \to \tilde H_k(X^{(n)}) \to \tilde H_k(\vee S^n)$$

we see $\tilde H_k(X^{(n-1)}) \to \tilde H_k(X^{(n)})$ is an isomorphism if $k \neq n-1, n$. If $k=n$ we get an injective map

$$\tilde H_n(X^{(n)}) \hookrightarrow \mathbb{Z}^{\#\text{ of } n\text{-cells}} \qquad (1)$$

and if $k=n-1$ we get

$$\mathbb{Z}^\ell \to \tilde H_{n-1}(X^{(n-1)}) \twoheadrightarrow \tilde H_{n-1}(X^{(n)}) \to 0 \qquad (2)$$

Then (a) is true by (1), (b) is true because (1) is an isomorphism and in (2) we have $\ell = 0$ giving

$$\begin{tikzcd} \tilde H_n(X^{(n+1)}) \arrow[d, phantom, sloped, "\cong"] & \tilde H_n(X^{(n)}) \arrow[l, "\cong"', "(2)"] \arrow[r, "\cong", "(1)"'] & \mathbb{Z}^{\#\, n\text{-cells}} \\ \tilde H_n(X) & & \end{tikzcd}$$

and (c) is true since in (2) $\tilde H_{n-1}(X^{(n-1)})$ has k generators, and $\tilde H_{n-1}(X^{(n)}) \cong \tilde H_{n-1}(X)$ is generated by their images.