Math 215B Homework 5

Due Friday February 29th, 2013 by 5 pm

Please remember to write down your name and Stanford ID number (9 digits). All pages and sections refer to pages and sections in Hatcher's *Algebraic Topology*.

1. (8 points) *The CW homology of the n-torus.* The n torus T^n, can be defined as

$$T^n = I^n/\sim, \quad I = [0,1] \tag{0.1}$$

where we identify $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \sim (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ for all i. To set notation, let e_i be the standard basis vectors of $\mathbb{R}^n$, and think of I^n as embedded in $\mathbb{R}^n$ in the usual way as

$$I^n = \{\vec{x} = x_1 e_1 + \cdots + x_n e_n | 0 \leq x_i \leq 1 \ \forall i\}. \tag{0.2}$$

Then any k and any subset $J = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ of size k, consider the k ball

$$B^k_J = I^n \cap \operatorname{span}(e_{i_1}, \ldots, e_{i_k}) \cong I^k \cong B^k. \tag{0.3}$$

Inclusion into I^n followed by projection gives a natural map

$$\Psi : \coprod_k \coprod_J B^k_J \longrightarrow T^n \tag{0.4}$$

These balls are the cells of a CW structure on $X = T^n$ as follows: define the k-skeleton of T^n to be

$$X^k := \Psi(\coprod_{i \leq k} \coprod_J B^i_J) \tag{0.5}$$

with X^{k+1} formed from X^k by all the attaching maps

$$\Psi : \coprod_J \partial B^{k+1}_J \longrightarrow X^k. \tag{0.6}$$

Note that with this cell structure, T^n has $\binom{n}{k}$ k cells for each k. Calculate the cellular homology by showing that the boundary maps in the cellular chain complex are all zero.

2. (5 points) *Homology with coefficients.* Using the standard CW structure, calculate the homology of $\mathbb{RP}^n$ with $\mathbb{Z}/6\mathbb{Z}$ coefficients.

3. (6 points) *Applications of degree.* Solve §2.2 (page 155), problem 2.

4. (10 points total) *The local degree of smooth maps.*
   a. (6 points) For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$, show that the **local degree of f at 0**, i.e. the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{Z}$, is 1 or -1 according to whether the determinant of f is positive or negative. [Hint: Use Gaussian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with ± 1's on the diagonal]

b. (4 points) Now let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a differentiable map, such that df_0, the total derivative matrix at 0, is invertible. Show that the local degree of f at 0 is once more 1 or -1 depending on whether the determinant of the derivative matrix is 1 or -1. (Hint: construct a homotopy of maps of pairs between f and df_0.)
Note: Our interest in this problem is as follows: By a general result, any map between smooth manifolds is homotopic to a smooth map. Thus, one way of computing the degree of a map $f : S^n \to S^n$ is:
- find a smooth map $\tilde{f}$ homotopic to f,
- apply Sard's theorem from differential topology to $\tilde{f}$ to find a point $q \in S^n$ with a finite collection of preimages $p_1, \ldots, p_k$ such that $d\tilde{f}_{p_i}$ is invertible for each i, and
- then, using this problem, perform the local degree calculation at p_i involving the sign of $\det(d\tilde{f}_{p_i})$ for each i.

5. (8 points) *Relating different definitions of degree.* Solve §2.2 (page 155), problem 8.

6. (6 points) *A computation in homology.* Solve §2.2 (page 156), problem 9b.

7. (10 points) *Computing the homology of a CW complex.* Solve §2.2 (page 156), problem 13.

8. (9 points total) *Euler characteristics.*
a. (3 points) A sequence of abelian groups $\{M_n\}$ is called **finite-dimensional** if there are finitely many non-zero M_i and if each M_i is finite rank. Given such a sequence, define its **algebraic Euler characteristic** as

$$\chi(\{M_n\}) = \sum_n (-1)^n \text{rank}(M_n). \tag{0.7}$$

Now, suppose we have a finite-dimensional chain complex (C_*, ∂_*). Prove that $\chi(\{C_n\}) = \chi(\{H_i(C_*, \partial)\})$.
b. (3 points) The **Euler characteristic** $\chi(X)$ of a finite CW complex X is defined as

$$\chi(X) = \sum_n (-1)^n c_n \tag{0.8}$$

where c_n is the number of n-cells of X (this generalizes the perhaps familiar formula *vertices - edges + faces*). It is not evident from this description, but in fact the Euler characteristic is a topological invariant of X. A definition that makes this more manifest (but perhaps harder to compute is)

$$\chi(X) := \sum_n (-1)^n \text{rank } H_n(X), \tag{0.9}$$

i.e. $\chi(X)$ is the algebraic Euler characteristic $\chi(\{H_i(X)\})$. Using results from earlier in this problem, prove that these two definitions of $\chi(X)$ coincide.
c. (3 points) Finally, prove that if $A, B \subset X$ are spaces whose interiors cover X, and the homology groups of X, A, B, and $A \cap B$ are all finite rank, then

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B). \tag{0.10}$$

Use this fact to compute $\chi(S^n \vee S^k)$.

9. (5 points total) *Obstructions to the existence of covering maps using Euler characteristics.*
   a. (3 points) Let X be a finite CW complex, and $\tilde{X} \to X$ an n-sheeted covering map. Prove that $\chi(\tilde{X}) = n\chi(X)$. (Namely, solve §2.2 (page 157), problem 22).
   b. (2 points) Use this to show that there is no finite covering map from a closed orientable surface of genus 6 to one of genus 3.

10. (6 points) *Homology of other complements in S^n.* Solve §2.B (page 176), problem 1.

11. (7 points) *The homology of a mapping torus.* Let X be a topological space, and $f : X \to X$ a homeomorphism. Recall that the **mapping torus** of f, denoted M_f, is defined to be theq quotient space of $X \times I$ by the relation $(x, 0) \sim (f(x), 1)$. (i.e., if f is the identity map, $M_f = X \times S^1$) Show how to use Mayer-Vietoris to compute the homology of the mapping torus, in terms of $H_*(Y)$ and f_*.