MATH 215B HW5

1. The attaching map for each k-cell

$$S^{k-1} \to X^{(k-1)} \to X^{(k-1)}/X^{(k-2)} \cong \vee S^{k-1} \to S^{k-1}$$

factors as

$$S^{k-1} \xrightarrow[\text{collapse all but two faces of cube}]{} S^{k-1} \vee S^{k-1} \xrightarrow[\text{id} \vee \text{id}]{} S^{k-1}$$

It is easy to see that the two pieces of the first map differ by a reflection in the domain, so they have degrees summing to 0. By the cellular boundary formula, the differentials of $C_*^{cw}(X)$ are therefore zero.

2. n even:

$$\begin{array}{cccccccccc} & C_n & & C_{n-1} & & & C_2 & & C_1 & & C_0 \\ & \mathbb{Z}/6 & \xrightarrow{2} & \mathbb{Z}/6 & \cdots & & \mathbb{Z}/6 & \xrightarrow{2} & \mathbb{Z}/6 & \xrightarrow{0} & \mathbb{Z}/6 \\ H_* = & \mathbb{Z}/2 & & \mathbb{Z}/2 & \cdots & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/6 \end{array}$$

n odd:

$$\begin{array}{cccccccccc} & \mathbb{Z}/6 & \xrightarrow{0} & \mathbb{Z}/6 & \cdots & & \mathbb{Z}/6 & \xrightarrow{2} & \mathbb{Z}/6 & \xrightarrow{0} & \mathbb{Z}/6 \\ H_* = & \mathbb{Z}/6 & & \mathbb{Z}/2 & \cdots & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/6 \end{array}$$

3. Either f has a fixed point, or

$f \sim$ antipodal map $\Rightarrow \deg f = -1$

$\Rightarrow \deg(-f) = 1$

$\Rightarrow -f$ has a fixed point.

If $g: \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ then $\exists$ a lift f

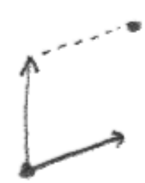
$$\begin{tikzcd} S^{2n} \arrow[r, dashed, "f"] \arrow[d, "p"] \arrow[dr, "g \circ p"] & S^{2n} \arrow[d, "p"] \\ \mathbb{RP}^{2n} \arrow[r, "g"] & \mathbb{RP}^{2n} \end{tikzcd}$$

by the π_1-lifting criterion. Picking $x \in S^{2n}$ s.t. $f(x) = x$ or $-x$, we see $p(x)$ is fixed under g.

Let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $90°$. Then $\phi^n: (\mathbb{R}^2)^n \to (\mathbb{R}^2)^n$ descends to a map $\mathbb{RP}^{2n-1} \circlearrowleft$ without fixed points $[x]$, since $\phi^n(x) \neq \pm x$.

4a. It suffices to show $GL_n(\mathbb{R})$ has two path components, $\det > 0$ and $\det < 0$. Visualizing each matrix by drawing its rows, we extend the row operations to continuous paths of invertible matrices:

multiply row by positive constant



add multiple of R1 to R2

R1 $\to$ R2

R2 $\to$ $-$R1

These generate all matrices of positive det so we are done.

4b. Use the straight-line homotopy from $f(x)$ to $df_0(x)$. For small ε, $\dfrac{|f(x)-df_0(x)|}{|x|} < \dfrac{1}{2|df_0^{-1}|}$ for $|x|<\varepsilon$

$$\Rightarrow |df_0^{-1}||f(x)-df_0(x)| < \frac{|x|}{2}$$

$$\Rightarrow |df_0^{-1}\circ f(x) - x| < \frac{|x|}{2}$$

so the straight line from x to $df_0^{-1}\circ f(x)$ misses 0, so the line from $df_0(x)$ to $f(x)$ misses 0 too. This gives a homotopy of pairs

$$(B_\varepsilon(0), B_\varepsilon(0)-\{0\}) \to (\mathbb{R}^n, \mathbb{R}^n-\{0\})$$

between f and df_0, so they have the same local degree.

5. Let z_0 be a root of f, and use an FLT to bring z_0 to the origin while preserving ∞. Now f looks like $z^r\prod(z-a_i)$ where r is the multiplicity of f at z_0. A straight-line homotopy gives a homotopy of maps of pairs $\boxed{\varepsilon < \min|a_i|}$

$$(B_\varepsilon, B_\varepsilon-\{0\}) \to (\mathbb{C}, \mathbb{C}-\{0\})$$

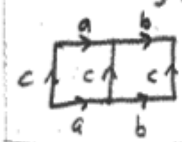
from our new f to $g(z)=z^r$. Now we just need $\deg_0 g = r$.

Compare two LES's:

$$\begin{array}{ccccccc}
\cancel{H_2(D^2)} & \to & H_2(D^2,S^1) & \xrightarrow{\sim} & H_1(S^1) & \to & \cancel{H_1(D^2)} \\
\downarrow & & \downarrow\wr & & \downarrow\wr & & \downarrow \\
\cancel{H_2(D^2)} & \to & H_2(D^2,D^2-0) & \xrightarrow{\sim} & H_1(D^2-0) & \to & \cancel{H_1(D^2)}
\end{array}$$

Clearly $g: S^1\to S^1$ has degree r, so under this identification $g: (D^2, D^2-\{0\}) \to (D^2, D^2-\{0\})$ has degree r.

6. Using Künneth or cellular homology,



$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$

$H_2 = \mathbb{Z}\oplus\mathbb{Z}$

7. $C_*^{cw} = \ldots 0 \to \mathbb{Z}^2 \xrightarrow{(2\ 3)} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

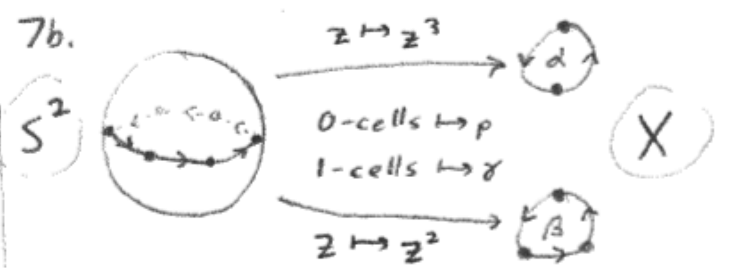
α,β $\quad\gamma\quad$ ρ

$\partial\alpha = 2\gamma$

$\partial\beta = 3\gamma$

A	$H_0 A$	$H_1 A$	$H_2 A$	$H_0\, X/A$	$H_1\, X/A$	$H_2\, X/A$
ρ	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\rho\cup\gamma$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$
$\rho\cup\gamma\cup\alpha$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\rho\cup\gamma\cup\beta$	$\mathbb{Z}$	$\mathbb{Z}/3$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\rho\cup\gamma\cup\alpha\cup\beta$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0

7b.



$$X \xrightarrow{/\gamma} \underset{\alpha}{S^2} \vee \underset{\beta}{S^2} \xrightarrow{id \vee id} S^2$$

These two maps $S^2 \to X \to S^2$ give the homotopy equivalence $S^2 \simeq X$, as checked laboriously or using Cor. 4.33 in Hatcher.

If $A = p \cup \gamma \cup \beta$ then

$$\begin{tikzcd} X \arrow[r] & X/A \arrow[d, "\simeq"] \\ S^2 \arrow[u, "\simeq"] \arrow[r, "\deg = 3"'] & S^2 \end{tikzcd}$$

and if $A = p \cup \gamma \cup \alpha$ then $X \to X/A$ is equivalent to $S^2 \xrightarrow[\deg = 2.]{} S^2$

8. a. By UCT we may tensor with $\mathbb{Q}$ so everything is a vector space. Then

$$\begin{aligned} \chi(H_i) &= \Sigma (-1)^i \text{ rank } H_i \\ &= \Sigma (-1)^i (\ker \partial_i - \operatorname{im} \partial_{i+1}) \\ &= \Sigma (-1)^i (\ker \partial_i + \operatorname{im} \partial_i) \\ &= \Sigma (-1)^i \text{ rank } C_i \\ &= \chi(C_i) \end{aligned}$$

b. The given definition is χ of the cellular chain complex, which by the above is $= \chi(\{H_i\})$

c. The Mayer-Vietoris sequence is exact $(\chi = 0)$. Therefore

$$\text{rank } H_0(A\cap B) - \text{rank } H_0 A - \text{rank } H_0 B + \text{rank } H_0 X - \text{rank } H_1(A \cap B) + \ldots = 0$$

$$\Sigma(-1)^i rk\, H_i(A\cap B) - \Sigma(-1)^i \left(rk\, H_i A + rk\, H_i B\right) + \Sigma(-1)^i rk\, H_i X = 0$$

$$\chi(A\cap B) - \chi(A) - \chi(B) + \chi(X) = 0$$

$$\begin{aligned} \chi(S^n \vee S^k) &= \chi(S^n) + \chi(S^k) - \chi(*) \\ &= (1 + (-1)^n) + (1 + (-1)^k) - 1 \\ &= 1 + (-1)^n + (-1)^k \end{aligned}$$

9. a. Put a CW structure on $\hat{X}$ with one k-cell for each lift of a k-cell in X. Then

$$\begin{aligned} \chi(\hat{X}) &= \Sigma(-1)^k \text{ rank } C_k^{cw}(\hat{X}) \\ &= \Sigma(-1)^k\, n \cdot \text{rank } C_k^{cw}(X) \\ &= n\, \chi(X) \end{aligned}$$

b. $\chi(M_3) = 1 - 6 + 1 = -4$

$\chi(M_6) = 1 - 12 + 1 = -10$

Since $\chi(M_3) \nmid \chi(M_6)$ there is no finite cover $M_6 \to M_3$.

10. Let $A = S^n - S^k$ and $B = S^n - S^\ell$. In the first case $(S^k \vee S^\ell)$ we have $A \cup B = S^n - * \simeq *$ so Mayer-Vietoris gives

$$\tilde{H}_i(A\cap B) \cong \tilde{H}_i(A) \oplus \tilde{H}_i(B) = \begin{cases} \mathbb{Z} & n-k-1 \\ \mathbb{Z} & n-\ell-1 \end{cases}$$

and $\mathbb{Z}^2$ in $n-k-1$ if $k = \ell$.

In the second case $(S^k \amalg S^\ell)$ we have $A \cup B = S^n$ so we get the above answer plus an extra $\mathbb{Z}$ in degree $n-1$.

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11. Cover $I = [0,1]$ by

$$U_1 = [0, \tfrac{1}{4}+\varepsilon) \cup (\tfrac{3}{4}-\varepsilon, 1]$$
$$U_2 = (\tfrac{1}{4}-\varepsilon, \tfrac{3}{4}+\varepsilon)$$

Let $A \subseteq M_f$ be the image of $X \times U_1$ and $B \subseteq M_f$ be the image of $X \times U_2$. Then

$$A \simeq X$$
$$B \simeq X$$
$$A \cap B \simeq X \amalg X$$
$$A \cup B = M_f$$

and under this equivalence

$$A \cap B \hookrightarrow A$$

becomes

$$X \amalg X \xrightarrow{\mathrm{id} \amalg \mathrm{id}} X$$

and

$$A \cap B \hookrightarrow B$$

becomes

$$X \amalg X \xrightarrow{f \amalg \mathrm{id}} X$$

so Mayer-Vietoris gives

$$H_n X \oplus H_n X \to H_n X \oplus H_n X \to H_n M_f \to \ldots$$

and this map has matrix

$$\begin{pmatrix} \mathrm{id} & -f_* \\ \mathrm{id} & -\mathrm{id} \end{pmatrix}$$