Math 215B Homework 6

Due Friday March 8th, 2013 by 5 pm

Please remember to write down your name and Stanford ID number (9 digits). All pages and sections refer to pages and sections in Hatcher's *Algebraic Topology*.

1. (8 points) *The Ham Sandwich theorem.* Prove the "ham sandwich theorem." That is, let f_1, f_2, and f_3 be three bounded, integrable functions on $\mathbb{R}^3$, each of which vanishes outside some compact set. Show that there is an affine plane Π cutting $\mathbb{R}^3$ into two half-spaces U^+ and U^-, such that

$$\int_{U^+} f_i(x,y,z)dxdydz = \int_{U^-} f_i(x,y,z)dxdydz, \quad \text{for } i = 1,2,3. \tag{0.1}$$

   Also, explain why this is called the "ham sandwich theorem".

2. (6 points) *Degree 1 cohomology classes.* Solve §3.1 (page 205), problem 5.

3. (10 points total) *Verifying UCT in an example.* As discussed in class, the tools developed for homology translate directly into the cohomological setting, up to reversing directions of arrows. For example, when X is a CW complex, excision and long exact sequences imply there is a small **cellular co-chain complex** $(C^*_{CW}(X,G), \delta_{CW})$ computing the cohomology of X with G coefficients, and the Universal Coefficient Theorem implies that the complex is the G-dual of the cellular chain complex,

$$C^i_{CW}(X,G) = \mathrm{Hom}(C_i^{CW}(X), G). \tag{0.2}$$

   (See §3.1, Theorem 3.5 on page 203 for a short proof spelling out the details).

   a. (2 points) Compute $\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$ for primes p, q directly from definitions.
   b. (8 points) Calculate $H^n(\mathbb{R}P^7, \mathbb{Z}/6\mathbb{Z})$, (i) starting from your knowledge of $H_n(\mathbb{R}P^7)$ the result in part (a), and then (ii) more directly from the cellular cochain complex.

4. (8 points) *Non-naturality of the splitting in Universal Coefficient Theorem.* Solve §3.1 (page 205), problem 11.

5. (8 points total) *Applications of UCT.*
   a. (4 points) Show that $H^1(X)$ is always torsion free, for any X.
   b. (4 points) Prove that, if the singular homology groups of X are finitely generated, then the singular cohomology of X admits the following description:

$$H^n(X) \cong F_n(X) \oplus T_{n-1}(X). \tag{0.3}$$

   Here $F_n(X)$ denotes the free subgroup of $H_n(X)$ and $T_{n-1}(X)$ denote the torsion subgroup of $H_{n-1}(X)$.

6. (5 points) *A computation using cup product.* Solve §3.2 (page 229), problem 6.

7. (5 points) *Distinguishing spaces using cup product.* Solve §3.2 (page 229), problem 7.

8. (5 points) *Cohomology rings with coefficients.* Solve §3.2 (page 229), problem 9.