$\boxed{\text{MATH 215B HW6}}$

1. For each $x \in S^2$ and $c \in \mathbb{R}$ let $P_{x,c}$ denote the plane $\{y : x \cdot y = c\}$ and

$$\begin{cases} U^+_{x,c} = \{y : x \cdot y > c\} \\ U^-_{x,c} = \{y : x \cdot y < c\} \end{cases}$$

Using standard analysis there is a unique largest $[a,b] \subseteq \mathbb{R}$ such that $\int_{U^+_{x,c}} f_1 = \int_{U^-_{x,c}} f_1$ for all $c \in [a,b]$. Let $c(x) = \frac{a+b}{2}$ and define $S^2 \xrightarrow{\varphi} \mathbb{R}^2$

$$x \longmapsto \left(\int_{U^+_{x,c(x)}} f_2, \int_{U^+_{x,c(x)}} f_3 \right)$$

By Borsuk-Ulam there exists an x for which $\varphi(x) = \varphi(-x)$. Then $P_{x,c(x)}$ is the desired hyperplane.

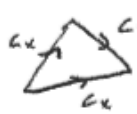
It's called the Ham Sandwich Thm. because eating a ham sandwich may help you prove it by giving you an energy boost.

2.(a)  $\xrightarrow{\sigma} X$

$\boxed{\varphi \in C^1(X;G), \ \delta\varphi = 0}$

$$\varphi(\partial\sigma) = \delta\varphi(\sigma) = 0(\sigma) = 0$$

$$\Rightarrow \varphi(f) + \varphi(g) - \varphi(f \cdot g) = 0$$

(b)  $\xrightarrow[c_x]{\text{constant}} X$

$$\Rightarrow \varphi(c_x) + \varphi(c_x) - \varphi(c_x) = 0$$

$$\Rightarrow \varphi(c_x) = 0$$

(c)  $\longrightarrow X \qquad f \simeq g$

$$\Rightarrow \varphi(f) + \cancel{\varphi(c_x)} - \varphi(g) = 0$$

$$\varphi(f) = \varphi(g)$$

(d) If $\varphi = \delta\gamma$, $\gamma \in C^0(X;G)$

then
$$\begin{aligned} \varphi(f) &= \delta\gamma(f) \\ &= \gamma(\partial f) \\ &= \gamma(f(1)) - \gamma(f(0)) \end{aligned}$$

so $\varphi(f)$ depends only on $f(0), f(1)$.

Conversely, if φ depends only on endpoints, pick a basepoint x_0 in each path component of X and define $\gamma(x) = \varphi(\text{any path } x_0 \to x)$

Then $\gamma \in C^0(X;G)$ is well-defined and

$$\begin{aligned} \delta\gamma(f) &= \gamma(\partial f) \\ &= \gamma(f(1)) - \gamma(f(0)) \\ &= \varphi(x_0 \to f(1)) - \varphi(x_0 \to f(0)) \\ &= \varphi(x_0 \to f(1)) + \varphi(f(0) \to x_0) \\ &= \varphi(f(0) \to f(1)) \\ &= \varphi(f) \qquad \forall \text{ paths } f \end{aligned}$$

$$\Rightarrow \delta\gamma = \varphi$$

3.(a)

$$\begin{aligned} &\operatorname{Ext}^1(\mathbb{Z}/p, \mathbb{Z}/q) \\ &= H^1(\operatorname{Hom}(\mathbb{Z} \xrightarrow{p} \mathbb{Z}, \mathbb{Z}/q)) \\ &= H^1(\mathbb{Z}/q \xleftarrow{p} \mathbb{Z}/q) \\ &= \operatorname{coker}(\mathbb{Z}/q \xleftarrow{p} \mathbb{Z}/q) \\ &\cong \begin{cases} \mathbb{Z}/q & \text{if } p = q \\ 0 & \text{if } p \neq q \quad (\text{prime}) \end{cases} \end{aligned}$$

(b)

	7	6	5	4	3	2	1	0
$H_*(\mathbb{RP}^7;\mathbb{Z}) \cong$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}$
$\mathrm{Hom}(H_*, \mathbb{Z}/6) \cong$	$\mathbb{Z}/6$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/6$
$\mathrm{Ext}(H_{*-1}, \mathbb{Z}/2) \cong$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	0
$\Rightarrow H^*(\mathbb{RP}^7;\mathbb{Z}/6) \cong$	$\mathbb{Z}/6$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/6$

$$\cong H^*\left(\mathbb{Z}/6 \xleftarrow{0} \mathbb{Z}/6 \xleftarrow{2} \mathbb{Z}/6 \xleftarrow{0} \cdots \xleftarrow{0} \mathbb{Z}/6\right)$$

4. (a) $X \to X/S^n$ is trivial on $\tilde{H}_*$ since for each i either $\tilde{H}_i(X) = 0$ or $\tilde{H}_i(X/S^n) = 0$. Therefore we get the 0 map on the summands

$$\mathrm{Hom}(H_{n+1}(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_n(X), \mathbb{Z})$$

but $H^{n+1}(X/S^n) \longrightarrow H^{n+1}(X)$ is

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z}/m$$

by comparing cochain complexes:

$$\begin{tikzcd}
\cdots & \mathbb{Z}_{(n+1)} \arrow[l] \arrow[d, "1"] & 0_{(n)} \arrow[l] \arrow[d] & \cdots \arrow[l] & C^*(X/S^n) \\
\cdots & \mathbb{Z}_{(n+1)} \arrow[l] & \mathbb{Z}_{(n)} \arrow[l, "m"'] & \cdots \arrow[l] & C^*(X)
\end{tikzcd}$$

so the splitting cannot be made natural.

(b) $S^n \hookrightarrow X$ is on chains

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & \cdots & C_*(S^n) \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r, "m"] & \mathbb{Z} \arrow[r] & \cdots & C_*(X)
\end{tikzcd}$$

so on homology it's nonzero:

$$\begin{tikzcd}
0 \arrow[d] & \mathbb{Z} \arrow[d, "1"] & \cdots & H_*(S^n) \\
0 & \mathbb{Z}/m & \cdots & H_*(X)
\end{tikzcd}$$

but on cochains we get

$$\begin{tikzcd}
\leftarrow 0 & \mathbb{Z} \arrow[l] & \arrow[l] & C^*(S^n) \\
\leftarrow \mathbb{Z} \arrow[u] & \mathbb{Z} \arrow[l, "m"'] \arrow[u, "1"'] & \arrow[l] & C^*(X)
\end{tikzcd}$$

$$\begin{tikzcd}
0 & \mathbb{Z} & H^*(S^n) \\
\mathbb{Z}/m \arrow[u] & 0 \arrow[u] & H^*(X)
\end{tikzcd}$$

so it's 0 on cohomology.

5. (a) $H^1(X) \cong \mathrm{Hom}(H_1(X), \mathbb{Z})$

$\oplus\, \mathrm{Ext}(H_0(X), \mathbb{Z})$

always torsion free

$\cong \mathrm{Ext}(\bigoplus_A \mathbb{Z}, \mathbb{Z}) = 0$

(b) $H_n(X) \cong F_n(X) \oplus T_n(X)$

$F_n(X) \cong \bigoplus_A \mathbb{Z}$

$T_n(X) \cong \bigoplus_i^k \mathbb{Z}/a_{i,n}$

$$\Rightarrow H^n(X) \cong \mathrm{Hom}(H_n(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(X), \mathbb{Z})$$
$$\cong \mathrm{Hom}(F_n(X), \mathbb{Z}) \oplus \mathrm{Ext}(T_{n-1}(X), \mathbb{Z})$$
$$\cong \mathrm{Hom}(\bigoplus_A \mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}(\bigoplus \mathbb{Z}/a_{i,n-1}, \mathbb{Z})$$
$$\cong \prod_A \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \prod \mathrm{Ext}(\mathbb{Z}/a_{i,n-1}, \mathbb{Z})$$
$$\cong \bigoplus_A \mathbb{Z} \oplus \bigoplus \mathbb{Z}/a_{i,n-1}$$
$$\cong F_n(X) \oplus T_{n-1}(X)$$

6. The d^{th} power map preserves the subspace $\mathbb{CP}^1 \subseteq \mathbb{CP}^n$. Identifying $(\mathbb{C}^2 - \{0\})/\mathbb{C}^\times \xrightarrow{\cong} \mathbb{C} \cup \{\infty\}$

$$(z_0, z_1) \longmapsto \left(\frac{z_0}{z_1}\right)$$

the d^{th} power map becomes $z \mapsto z^d$, which from last week has degree d. If $\alpha \in H^2(\mathbb{C}P^n)$ is a generator then the square

$$\begin{array}{ccc} H^2(\mathbb{C}P^n) & \longrightarrow & H^2(\mathbb{C}P^n) \\ \downarrow \cong & & \downarrow \cong \\ H^2(\mathbb{C}P^1) & \xrightarrow{\cdot d} & H^2(\mathbb{C}P^1) \end{array}$$

forces us to conclude α goes to $d\alpha$, and therefore $\alpha^k \in H^{2k}(\mathbb{C}P^n)$ goes to $d^k \alpha^k$.

7. $H^*(\mathbb{R}P^3; \mathbb{Z}/_2) \cong \mathbb{Z}[\alpha]/_{\alpha^4}, \quad |\alpha| = 1$

$H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}/_2) \cong \mathbb{Z}[\beta, \gamma]/_{\beta^3, \beta\gamma, \gamma^2}$

$|\beta| = 1, \ |\gamma| = 3$

It's easy to check there is no isomorphism of graded rings between these.

8. $H^n(X; \mathbb{Z}) \to H^n(X; \mathbb{Z}/_p)$ is always a map of rings that factors through $H^n(X;\mathbb{Z}) \otimes \mathbb{Z}/_p$. When $H_i(X)$ is free, the square

$$\begin{array}{ccc} H^n(X;\mathbb{Z}) \otimes \mathbb{Z}/_p & \longrightarrow & H^n(X; \mathbb{Z}/_p) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(H_n(X), \mathbb{Z}) \otimes \mathbb{Z}/_p & \longrightarrow & \mathrm{Hom}(H_n(X), \mathbb{Z}/_p) \\ \downarrow \cong & & \downarrow \cong \\ (\prod_A \mathbb{Z}) \otimes \mathbb{Z}/_p & \xrightarrow{\cong} & \prod_A \mathbb{Z}/_p \end{array}$$

shows the top map is an isomorphism. (The bottom isom. is not obvious and requires some care to prove injectivity when the product is infinite.)