# Math 215B suggested exercises, after Homework 6

1. *The cohomology ring of a genus $g$ surface.* Solve §3.2 (page 228), problem 1.

2. *Restrictions on induced maps coming from cup product.* Solve §3.2 (page 229), problem 11.

3. *Non-equivalent spaces with isomorphic cohomology rings.* Solve §3.2 (page 229), problem 12.

4. *Orientability is unaffected by removing points.* Solve §3.3 (page 257), problem 2.

5. *The degree of maps between manifolds.* For a map $f : M \to N$ between connected closed orientable $n$-manifolds with fundamental classes $[M]$ and $[N]$, the **degree** of $f$ is defined to be the integer $d$ such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes.
   a. *A map to $S^n$ of degree 1.* Solve §3.3 (page 258), problem 7.

   b. *The degree of a covering map.* Solve §3.3 (page 258), problem 9.

   c. *The effect of degree 1 maps on $\pi_1$.* Solve §3.3 (page 258), problem 10.

6. *Homology as a module over cohomology.* Show that $(\alpha \cap \phi) \cap \psi = \alpha \cap (\phi \cup \psi)$ for all $\alpha \in C_k(X; R)$, $\phi \in C^l(X; R)$, and $\psi \in C^m(X; R)$. Deduce that cap product makes $H_*(X; R)$ a right $H^*(X; R)$ module.

7. *The homology groups of 3-manifolds.* Solve the first part of §3.3 (page 259), problem 24. Namely: let $M$ be a closed, connected 3-manifold, and write $H_1(M; R)$ as $\mathbb{Z}^r \oplus T$, the direct sum of a free abelian group of rank $r$ and a finite group $T$. Show that $H_2(M; \mathbb{Z})$ is $\mathbb{Z}^r$ if $M$ is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$ if $M$ is non-orientable. In particular, $r \geq 1$ when $M$ is nonorientable.

8. *The compactly supported cohomology of a product with $\mathbb{R}$.* Unlike ordinary cohomology, compactly supported cohomology is *not* a homotopy invariant. As an instance of this fact, prove that
$$H_c^n(X \times \mathbb{R}; G) \simeq H_c^{n-1}(X; G)$$
for all $n$ (the reader should be reminded of taking homology or cohomology of a *suspension*).