Math 215B Final Exam — Solutions

Held Thursday, March 21, 2013, 7 - 10 pm.

1. (20 points total) For this question, assume if necessary that all spaces are path connected, locally path connected, and semi-locally simply connected.

   a. (10 points) Suppose X is a space, and let $Y \subset X$ be a path-connected subspace. Let $p : \tilde{X} \to X$ denote the universal cover, and suppose that the pre-image of Y
   $$\tilde{Y} := p^{-1}(Y)$$
   is path-connected. Prove that the map induced by inclusion $i_* : \pi_1(Y) \to \pi_1(X)$ is surjective.

   b. (10 points) Let X_1 and X_2 be spaces, with $\pi_1(X_i) = G_i$ for $i = 1, 2$. Find a group-theoretic condition on the group $G_1 \times G_2$ which is equivalent to the condition that every cover of $X_1 \times X_2$ is a product of a cover of X_1 with a cover of X_2. If $G_1 = G_2 = \mathbb{Z}/2\mathbb{Z}$, is every cover of $X_1 \times X_2$ a product cover?

Solution: **1a.** Let γ be a loop in X based at x_0. Choose a preimage $\widetilde{x}_0$ of x_0. Then there is a unique lift $\widetilde{\gamma}$ in $\widetilde{X}$ that starts at $\widetilde{x}_0$ and ends at some other preimage $\widetilde{x}_1$. Both $\widetilde{x}_0$ and $\widetilde{x}_1$ are in $p^{-1}(x_0) \subset \widetilde{Y}$. Since $\widetilde{Y}$ is path-connected, there is another path α from $\widetilde{x}_0$ to $\widetilde{x}_1$ that lies entirely in $\widetilde{Y}$. Since $\widetilde{X}$ is simply-connected, there is a homotopy from $\widetilde{\gamma}$ to α that preserves the endpoints. Applying p, we get a homotopy from γ to $p(\alpha) \subset Y$, so γ is homotopic to α, which is in the image of $\pi_1(Y)$. Therefore the map
$$\pi_1(Y) \to \pi_1(X)$$
is surjective.

1b. The group-theoretic condition is that every subgroup of $G_1 \times G_2$ is a product subgroup $H_1 \times H_2$ for subgroups $H_i \leq G_i$. If this condition holds, then every covering space
$$\widetilde{Y} \to X_1 \times X_2$$

is associated to a product subgroup $H_1 \times H_2$. Taking a cover $\widetilde{X}_i \to X_i$ corresponding to H_i and taking the product, we get a cover

$$\widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$$

which hits the same subgroup as $\widetilde{Y}$. Therefore $\widetilde{Y} \cong \widetilde{X}_1 \times \widetilde{X}_2$ as covering spaces. So every covering space is isomorphic to a product of covers of X_1 and X_2.

Conversely, if every cover is isomorphic to a product, then given a subgroup $K \leq G_1 \times G_2$, we build the cover corresponding to K, which must be of the form

$$\widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$$

The image of π_1 is then

$$K = p_* \pi_1(\widetilde{X}_1) \times p_* \pi_1(\widetilde{X}_2) \leq G_1 \times G_2$$

so K is a product subgroup.

Finally, $\mathbb{Z}/2 \times \mathbb{Z}/2$ does not satisfy this condition because the subgroup generated by $(1, 1)$ is not a product subgroup.

2. (15 points) Let $X = \mathbb{R}P^2 \vee S^3$ and $Y = \mathbb{R}P^3$. Prove that the homology and cohomology groups of X and Y are isomorphic with any coefficients, but that X and Y do not have the same homotopy type.

2. Take cellular chains with G coefficients:

$$\begin{array}{cccccccccc}
X & & & C_3^{CW}(X;G) & & C_2^{CW}(X;G) & & C_1^{CW}(X;G) & & C_0^{CW}(X;G) \\
\mathbb{RP}^2 \vee S^3 & \dots & \longrightarrow & G & \xrightarrow{0} & G & \xrightarrow{2} & G & \xrightarrow{0} & G \\
\mathbb{RP}^3 & \dots & \longrightarrow & G & \xrightarrow{0} & G & \xrightarrow{2} & G & \xrightarrow{0} & G
\end{array}$$

These chain complexes are isomorphic, so they give isomorphic homology groups. Similarly, the cochain complexes are isomorphic:

$$\begin{array}{cccccccccc}
X & & & C^3_{CW}(X;G) & & C^2_{CW}(X;G) & & C^1_{CW}(X;G) & & C^0_{CW}(X;G) \\
\mathbb{RP}^2 \vee S^3 & \dots & \longleftarrow & G & \xleftarrow{0} & G & \xleftarrow{2} & G & \xleftarrow{0} & G \\
\mathbb{RP}^3 & \dots & \longleftarrow & G & \xleftarrow{0} & G & \xleftarrow{2} & G & \xleftarrow{0} & G
\end{array}$$

So the cohomology groups are isomorphic with any coefficients.

If $\mathbb{RP}^2 \vee S^3$ were homotopy equivalent to $\mathbb{RP}^3$, then their cohomology with $\mathbb{Z}/2$ coefficients would define isomorphic rings, but we've seen on the homework that is not the case.

3. (10 points) Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

   3. The quotient map $S^1 \times S^1 \to (S^1 \times S^1)/(S^1 \vee S^1)$, on second homology, fits into the LES of the pair $(S^1 \times S^1, S^1 \vee S^1)$:

   $$0 \to H_2(S^1 \times S^1) \to H_2((S^1 \times S^1)/(S^1 \vee S^1)) \xrightarrow{0} H_1(S^1 \vee S^1) \xrightarrow{\sim} H_1(S^1 \times S^1) \to \dots$$

   The above map of H_1s is an isomorphism, so we can conclude that the above map on H_2 is an isomorphism. Therefore the quotient map is not nullhomotopic.

   On the other hand, the π_1-criterion guarantees that any map $S^2 \to S^1 \times S^1$ lifts to the universal cover $\mathbb{R} \times \mathbb{R}$, where it is homotopic to a constant map. Projecting this homotopy back down, we see our original map was nullhomotopic.

4. (10 points) Let M be a compact, connected, oriented n dimensional manifold, and let

   $$h^k(M) := \frac{H^k(M)}{\text{Torsion}(H^k(M))}$$

   denote the torsion free subgroup of singular cohomology of M. If μ_M is a fundamental class of M, one can construct a pairing:

   $$\begin{aligned} H^k(M) \times H^{n-k}(M) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto (\alpha \cup \beta) \cap \mu_M. \end{aligned}$$

   One consequence of Poincaré duality is that when restricted to torsion free parts, this pairing is is **unimodular**, meaning that the induced map

   $$h^k(M) \to \hom(h^{n-k}(M), \mathbb{Z})$$

   is an isomorphism. This follows from Poincaré duality and the relationship between cup and cap product

   $$(\alpha \cup \beta) \cap \mu_M = \alpha \cap (\beta \cap \mu_M).$$

Now, let M be a compact, connected, oriented 3-dimensional manifold, with

$$\pi_1(M) = \mathbb{Z}.$$

Determine the homology and cohomology groups of M, and the ring structure on cohomology, all with $\mathbb{Z}$ coefficients. (Hint: don't forget that the cup product ring is *graded commutative*, i.e. $a \cup b = (-1)^{qp} b \cup a$ if $a \in H^p(M)$, $b \in H^q(M)$).

Solution: M is connected, so $H_0(M) = \mathbb{Z}$, and by the Universal Coefficient Theorem (UCT), $H^0(M) = \mathbb{Z}$. Next, we are given that $\pi_1(M) = \mathbb{Z}$, so since H_1 is the abelianization of π_1, this implies $H_1(M) = \mathbb{Z}$. By UCT, since $H_0(\mathbb{Z})$ is free, $H^1(M) = \mathbb{Z}$.

Now, we turn to Poincaré duality. Since M is an oriented connected, compact 3-manifold, duality implies that

$$H_2(M) \cong H^1(M) = \mathbb{Z} \tag{1}$$
$$H_3(M) \cong H^0(M) = \mathbb{Z} \text{ (or because } M \text{ is oriented)} \tag{2}$$
$$H^2(M) \cong H_1(M) = \mathbb{Z} \tag{3}$$
$$H^3(M) \cong H_0(M) = \mathbb{Z}, \tag{4}$$
$$H^{i+3}(M) \cong H_{-i}(M) = 0, \; i > 1 \tag{5}$$
$$H_{i+3}(M) \cong H^{-i}(M) = 0, \; i > 1. \tag{6}$$

The last two facts about vanishing of cohomology/homology in degrees greater than three can be seen more directly from Hatcher's theorem about the homology of oriented manifolds followed by applying UCT to get cohomology (Theorem 3.26).

Recapping, we see that

$$H_i(M) \cong H^i(M) = \begin{cases} \mathbb{Z} & i = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

In particular, since each H^i is torsion free, $h^i(M) = H^i(M)$.

Finally, we need to compute the ring structure on $H^*(M)$. We know this ring has a unit, so the generators in degree zero act as multiplication by ± 1. Let's determine the other products.

Pick generators α, β, γ for $H^1(M) \simeq \mathbb{Z}$, $H^2(M) \simeq \mathbb{Z}$, and $H^3(M) \simeq \mathbb{Z}$ respectively. Any product with degree greater than three vanishes, e.g. $\gamma \cup \gamma = 0$, so the only possibly non-zero products are $\alpha \cup \alpha$, $\alpha \cup \beta$.

Let μ_M denote a fundamental class of M. In degree 3, Poincaré duality implies $\gamma \cap \mu_M = 1$ (otherwise replace γ with $-\gamma$), so capping with μ_M is the unique isomorphism $H^3(M) \to \mathbb{Z}$ sending γ to 1. Unimodularity of the cup product pairing then implies that given a generator α of $H^1(M)$, there is a generator $\beta' = \pm\beta$ of $H^2(M)$ such that $\alpha \cup \beta' = \gamma$, because the map

$$\begin{aligned} H^1(M) &\longrightarrow \hom(H^2(M), \mathbb{Z}) \\ x &\mapsto (\cdot \cup x) \cap \mu_M \end{aligned} \tag{8}$$

is an isomorphism, and in particular sends a generator to the generating homomorphism $\check{\beta} : H^2(M) \to \mathbb{Z}$ mapping β to 1. Replacing β by $-\beta$ if necessary, this implies that $\alpha \cup \beta = \gamma$.

It remains only to determine $\alpha \cup \alpha$ in terms of β. Since α lives in degree 1, by graded commutativity of the cup product

$$\alpha \cup \alpha = (-1)^{1 \cdot 1} \alpha \cup \alpha,$$

so

$$2\alpha \cup \alpha = 0.$$

Since $\alpha \cup \alpha \in H^2(M) = \mathbb{Z}$, which is an integral domain, this implies

$$\alpha \cup \alpha = 0.$$

Hence, we see that the cohomology ring of M is

$$H^*(M) = \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2) \tag{9}$$

with α in degree 1, β in degree 2.

5. (10 points) Is $(S^2 \times S^4) \vee S^8$ homeomorphic to a compact closed manifold? Explain.

Solution: Originally, this problem was supposed to say *homotopy-equivalent* instead of *homeomorphic*, but in light of the typo, we accepted any solutions that proved that $(S^2 \times S^4) \vee S^8$ was not homeomorphic (which is weaker than being non-homotopy equivalent). In this solution set, we will show both. Let $X = (S^2 \times S^4) \vee S^8$.

X **is not homeomorphic to a manifold**: Let p denote the wedge point. If X was homeomorphic to a manifold, a neighborhood U of p would be homeomorphic to $\mathbb{R}^n$, meaning that $U - p$ should be homeomorphic to $\mathbb{R}^n - p$. But $U - p$ is disconnected, whereas $\mathbb{R}^n - p$ is connected for $n > 1$. To see that $n = 1$ is not a possible dimension for X, note that a point on S^8 away from the wedge point admits neighborhoods homeomorphic to $\mathbb{R}^8$, and $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ if $m \neq n$ (this was proven in Hatcher/class—apply invariance of domain).

Alternatively, take a point on S^8 and a point on $S^2 \times S^4$, both away from the wedge, and note that they admit neighborhoods locally homeomorphic to $\mathbb{R}^8$ and $\mathbb{R}^6$ respectively, a contradiction for the same reasons.

X **is not homotopy-equivalent to a manifold**. In class, we showed that the homology of $S^2 \times S^4$ admits a cell structure with one 0 cell, 2 cell, 4 cell, and 6 cell, so X admits a cell structure with all of these and an 8 cell. Thus by cellular homology and cohomology, the homology/cohomology of X with $\mathbb{Z}/2$ coefficients is $\mathbb{Z}/2$ in degrees 0, 2, 4, 6, 8, and zero otherwise. Thus, if X were homotopy equivalent to a compact connected manifold M, since all manifolds are $\mathbb{Z}/2$ orientable, we conclude it would have to be of dimension 8. Moreover, since the 8th homology of X with $\mathbb{Z}$ coefficients is $\mathbb{Z}$, M would have to be orientable.

Now, we will show that the induced pairing on the cohomology ring of X is *not* unimodular. Firstly, the cohomology ring of $S^2 \times S^4$ is by Künneth $H^*(S^2) \otimes H^*(S^4) = \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ where α has degree 2, β has degree 4. Let $p_1 : X \to S^2 \times S^4$, $p_2 : X \to S^8$ denote the projections and $i_1 : S^2 \times S^4 \to X$, $i_2 : S^8 \to X$ the inclusions. We have that $p_1 \circ i_1 = id_{S^2 \times S^4}$ and $p_2 \circ i_2 = id_{S^8}$, so both S^8 and $S^2 \times S^4$ are retracts of X. The cohomology groups of X are $\mathbb{Z}$ in degrees 0, 2, 4, 6, 8, where the degree 2,4 and 6 classes are $i_1^*(\alpha)$, $i_1^*(\beta)$, $i_1^*(\alpha \cup \beta)$, and the degree 8 class is $i_2^*(\gamma)$, where $\gamma \in H^8(S^8)$ is a generator.

Note that given $i_1^*(\alpha \cup \beta)$ in degree 6, there is no element of degree 2 that gives a non-zero cup product. This is because all elements of degree 2 are of the form $d \cdot i_1^*(\alpha)$, and by naturality

$$d \cdot i_1^*(\alpha) \cup i_1^*(\alpha \cup \beta) = d \cdot i_1^*(\alpha \cup \alpha \cup \beta) = 0.$$

This violates Poincaré duality, which would imply that there exists an element $\kappa \in H^2(X)$ such that $i_1^*(\alpha \cup \beta) \cup \kappa$ generates $H^8(X) = \mathbb{Z}$. This concludes the proof.

6. (15 points) Compute the cohomology ring $H^*(T \times K; \mathbb{Z}/2\mathbb{Z})$, where T denotes the 2-torus and K denotes the Klein bottle (Hint: the cup product can be computed directly from definitions using a Δ-complex structure.)

   Solution: see next page.

6). For $T^2 = S^1 \times S^1$, note that by Künneth (since S^1 is CW & $H^*(S^1; \mathbb{Z}/2)$ is free & finitely generated),

$$H^*(T^2; \mathbb{Z}/2) \cong (H^*(S^1; \mathbb{Z}/2)) \otimes (H^*(S^1; \mathbb{Z}/2))$$

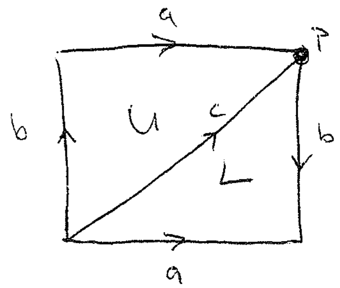
so

$$H^i(T^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\langle e\rangle & i=0 \\ \mathbb{Z}/2\langle\alpha\rangle + \mathbb{Z}/2\langle\beta\rangle & i=1 \\ \mathbb{Z}/2\langle\alpha\beta\rangle & i=2. \end{cases}$$

with $\alpha\beta = -\beta\alpha = \beta\alpha$ (as $-1 = 1$ in $\mathbb{Z}/2$), so

$$H^*(T^2; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha,\beta]/\alpha^2, \beta^2.$$

For K, we write out the following Δ-complex:



2-simplices are oriented so $e_0 \to e_1 \to e_2$ the edges match the arrows on the edges, so

$\partial L = b - a + c$ in that order, i.e.

$L|_{[e_0,e_1]} = c$, $L|_{[e_1,e_2]} = b$

$U|_{[e_0,e_1]} = b$, $U|_{[e_1,e_2]} = a$. (this is important!)

$$C^i_\Delta(K, \mathbb{Z}/2) = \mathrm{Hom}(C_i^\Delta(K); \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\langle p^\vee\rangle & i=0 \\ \mathbb{Z}/2\langle a^\vee\rangle + \mathbb{Z}/2\langle b^\vee\rangle + \mathbb{Z}/2\langle c^\vee\rangle & i=1 \\ \mathbb{Z}/2\langle U^\vee\rangle + \mathbb{Z}/2\langle L^\vee\rangle & i=2. \end{cases}$$

where $a^\vee$ e.g. is the dual of a, so $a^\vee(a) = 1$, $a^\vee(b) = 0$, $a^\vee(c) = 0$.

The coboundary acts as:

deg 0:

$\delta p^\vee(a) = p^\vee(\partial a) = 0$ $\qquad \delta p^\vee(c) = p^\vee \partial c$

$\delta p^\vee(b) = p^\vee(\partial b) = 0$

deg 1: $\delta a^\vee(U) = a^\vee(\partial U) = a^\vee(\pm a \pm b \mp c) = \pm 1 = 1 \mod 2$.

$\delta a^\vee(L) = 1 \mod 2$

Similarly, $\delta b^\vee(U) = 1 \mod 2$, $\delta b^\vee(L) = 1 \mod 2$

and $\delta c^\vee(U) = \delta c^\vee(L) = 1 \mod 2$.

so $\delta a^\vee = \delta b^\vee = \delta c^\vee = U^\vee + L^\vee \mod 2$.

deg 2:

$\delta U^\vee = \delta L^\vee = 0$.

Thus, the chain complex is

$$0 \to \mathbb{Z}/2 \xrightarrow{0} (\mathbb{Z}/2)^3 \longrightarrow (\mathbb{Z}/2)^2 \longrightarrow 0$$

$i = 0$ $\qquad (a^\vee, b^\vee, c^\vee) \longmapsto (u^\vee + L^\vee, u^\vee + L^\vee, u^\vee + L^\vee)$.

$\Rightarrow$ the cohomology $H^i(K; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, \text{ gen. by } p^\vee & i = 0 \\ (\mathbb{Z}/2)^2, \text{ gen. by 2 out of } (a^\vee + b^\vee), (a^\vee + c^\vee), (b^\vee + c^\vee) & i = 1 \\ \mathbb{Z}/2, \text{ gen. by } u^\vee \text{ or } L^\vee & i = 2. \end{cases}$

Pick representative cocycles for generators $[\alpha] = [a^\vee + b^\vee]$, $[\beta] = [a^\vee + c^\vee]$ for H^1,
$[u^\vee]$ for H^2.

Clearly, as cupping with $p^\vee$ is identity, and $[u^\vee] \smile [u^\vee] = 0$ in too high a degree, it suffices to compute cup products of these 1 cocycles.

In order for a cup product to not be zero on H^2, as a cocycle it must be zero on exactly one of U, L. (but not both).

Let's compute:

$$(\alpha \smile \alpha)(U) = \alpha(U|_{[e_0 e_1]})\, \alpha(U|_{[e_1 e_2]})$$
$$= \alpha(b)\, \alpha(a) = 1$$

$$\alpha \smile \alpha(L) = \alpha(L|_{[e_0 e_1]})\, \alpha(L|_{[e_1 e_2]}) = \alpha(c)\, \alpha(b) = 0$$

$$\alpha \smile \beta(U) = \alpha(U|_{[e_0 e_1]})\, \beta(U|_{[e_1 e_2]}) = \alpha(b)\, \beta(a) = 1$$

$$\alpha \smile \beta(L) = \alpha(L|_{[e_0 e_1]})\, \alpha(L|_{[e_1 e_2]}) = \alpha(c)\, \beta(b) = 0$$

$$\beta \smile \beta(U) = \beta(b)\, \beta(a) = 0$$

$$\beta \smile \beta(L) = \beta(c)\, \beta(b) = 0$$

Thus, letting $\hat{\alpha} = [\alpha]$, $\hat{\beta} = [\beta]$, $\gamma = [\hat{u}]$, we have that $\hat{\alpha}^2 = \hat{\alpha}\hat{\beta} = \gamma$, $\hat{\beta}^2 = 0$.

So $H^*(K; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha, \beta] / \alpha^3, \beta^2, \alpha^2 - \alpha\beta$.

Finally, by Künneth, as T^2 is a CW complex &
$H^*(T^2; \mathbb{Z}/2)$ is finitely generated & free over $\mathbb{Z}/2$,

$$H^*(K \times T; \mathbb{Z}/2) \cong H^*(K; \mathbb{Z}/2) \otimes H^*(T^2; \mathbb{Z}/2)$$

as rings. ∎