Math 215B Take-home Midterm Solutions

February 20, 2013

1. (10 points total) *Wrong-way maps.* We have seen that singular homology is a *functorial* assignment, that is, given a map $f : X \to Y$ of topological spaces, there is an induced map $f_* : H_i(X) \to H_i(Y)$ on homology groups. In some cases, if the map $f : X \to Y$ is particularly nice, there also exists a map $f^! : H_i(Y) \to H_i(X)$, called a *wrong-way* or *transfer* map.

   a. (5 points) Let $p : \tilde{X} \to X$ be a k-sheeted covering map, for some finite k. Construct a (non-trivial!) map of chain complexes

   $$C_i(X) \longrightarrow C_i(\tilde{X}) \tag{1}$$

   and show that it is a chain map, giving rise to an induced map on homology

   $$p^! : H_i(X) \longrightarrow H_i(\tilde{X}). \tag{2}$$

   b. (5 points). Show that the composition

   $$p_* \circ p^! : H_i(X) \to H_i(X) \tag{3}$$

   is multiplication by k.

Solution: **1a.** Define the chain map $\phi : C_n(X) \to C_n(\widetilde{X})$ by taking each n-simplex

$$\sigma : \Delta^n \to X$$

to the sum of its k lifts to $\widetilde{X}$. There are always exactly k lifts, since 1.33 and 1.34 in Hatcher tell us that each preimage of $\sigma(x_0)$ corresponds to a unique lift, and there are k such preimages. Taking the ith face of each of these k lifts, we get k distinct lifts of $\partial_i\sigma$, which must be the k unique lifts of this $(k-1)$-simplex. Therefore taking the sum of lifts commutes with ∂_i, so it commutes with $\partial = \sum_i (-1)^i \partial_i$ as well. Therefore ϕ a chain map, so it passes to a transfer map on homology

$$H_n(X) \to H_n(\widetilde{X})$$

1b. It suffices to show that $p_\# \circ \phi$ is multiplication by k, before passing to homology classes. Under this chain map, the simplex σ goes to the sum of its k lifts, each of which is then mapped back to σ, so we get a sum of k copies of σ. So $p_\# \circ \phi$ is multiplication by k and we are done.

2. (14 points total) *Applications to group theory.*

a. (7 points) Let X be a wedge sum of n circles, with its natural graph structure, and let $\tilde{X} \to X$ be a covering space with $Y \subset \tilde{X}$ a finite connected graph. Show there is a finite graph $Z \supset Y$ having the same vertices as Y, such that the projection $Y \to X$ extends to a covering space $Z \to X$.

b. (7 points) Using the above fact if necessary, prove the following result in group theory: Let F be a finitely generated free group, $H \subset F$ a finitely generated subgroup, and $x \in F - H$. Then there is a subgroup K of finite index such that $K \supset H$ and $x \notin K$.

Solution: **2a.** Label the n loops of X by $a_1, \ldots, a_n$. We will adopt the convention that each lift of a_i in $\widetilde{X}$ or Z will also be labelled a_i.

Suppose that Y has m vertices, and fix a value of i between 1 and n. There are k edges labelled a_i in Y, with $0 \leq k \leq m$. Since Y is contained in a cover of X, each vertex of Y has at most one edge labelled a_i coming in or going out. There are m vertices and k edges labelled a_i, so exactly $m - k$ of the vertices have no a_i coming in, and exactly $m - k$ of the vertices have no a_i going out. Therefore we may pick a bijection

$$\{\text{vertices with no } a_i \text{ going out}\} \xrightarrow{\cong} \{\text{vertices with no } a_i \text{ coming in}\}$$

and this bijection tells us how to attach $m - k$ more edges to Y so that each each vertex has exactly one edge labelled a_i pointing in and one edge labelled a_i pointing out. Doing this separately for each value of i, we arrive at a graph Z containing Y whose edges are labelled in a way that describes a covering map $Z \to X$ extending $Y \to X$. Note that Z will not in general be contained in $\widetilde{X}$.

2b. We are given a finitely generated free group F, a finitely generated subgroup H, and an element $x \in F - H$. Let X be a wedge of one circle for each generator of F, so $\pi_1(X) \cong F$. Let $(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a cover associated to $H \subset F$. For each element of some finite set of generators for H, pick a finite edgepath γ_i that begins and ends at $\widetilde{x}_0$. In addition, pick a finite edgepath α in $\widetilde{X}$ that starts at $\widetilde{x}_0$ and lifts the loop in X corresponding to x. Since x is not in the subgroup H,

α will not end at $\widetilde{x}_0$. Now let Y be the union of α and all the γ_i, and extend Y to a covering space Z using the above problem. Let $K \subset F$ be the subgroup corresponding to the image of $\pi_1(Z, \widetilde{x}_0)$ in $\pi_1(X)$. Then K obviously contains H, but it does not contain x because the path representing x in X lifts to α in Z, which is not a closed loop. Finally, K has finite index because Z has finitely many vertices, which are the preimages of the basepoint of X.

3. (12 points) *Spaces not distinguished by homology.* Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

   Solution: Using previous computations, the homology groups of $S^1 \times S^1$ are $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$. Using the formula for the reduced homology of a wedge, the homology groups of $S^1 \vee S^1 \vee S^2$ are also $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$. The universal cover of $S^1 \times S^1$ is $\mathbb{R}^2$, which is contractible, so it has the homology of a point. The universal cover of $S^1 \vee S^1 \vee S^2$ is obtained from the universal cover of $S^1 \vee S^1$ by attaching a copy of S^2 to every vertex. This is a 2-dimensional CW complex in which the 1-skeleton is a tree. Contracting this tree to a point, we get a countable wedge of 2-spheres, so its homology is a countable direct sum $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ in degree 2, and 0 in all other positive degrees.

4. (10 points) *Homological algebra.* Let (C^n_*, ∂) be a collection of chain complexes indexed by $n \in \mathbb{Z}$, i.e., for each $n \in \mathbb{Z}$, there is a chain complex

$$\cdots \to C^n_k \xrightarrow{\partial} C^n_{k-1} \xrightarrow{\partial} C^n_{k-2} \to \cdots . \tag{4}$$

Let $f^n_* : C^n_* \to C^{n+1}$ be a chain map, one for each n. Suppose that the composite $f_{n+1} \circ f_n : C^n \to C^{n+2}$ is chain-homotopic to zero for all n, by a chain homotopy $K^n : C^n_* \to C^{n+2}_{*+1}$; that is,

$$f^{n+1} \circ f^n = \partial K^n + K^n \partial \tag{5}$$

First part: Show that the map

$$\psi^n := f^{n+2} \circ K^n - K^{n+1} \circ f^n \tag{6}$$

is an *anti-chain map* from $C^n_* \to C^{n+3}_*$, meaning that $\partial \circ \psi^n = -\psi^n \circ \partial$, and deduce that ψ^n gives rise to a map on homology,

$$\psi^n_* : H_i(C^n_*, \partial) \longrightarrow H_{i+1}(C^{n+3}_*, \partial) \tag{7}$$

for all n and i. **Second part**: Finally, suppose that (7) is an isomorphism for all n and i. Deduce that the sequence

$$\cdots \longrightarrow H_i(C^n_*, \partial) \xrightarrow{f^n_*} H_i(C^{n+1}_*, \partial) \xrightarrow{f^{n+1}_*} H_i(C^{n+2}, \partial) \longrightarrow \cdots \tag{8}$$

is exact.

Solution of first part: First, let's check that ψ^n is an anti-chain map. We compute, using the chain homotopy equation $f^{n+1} \circ f^n = \partial K^n + K^n \partial$ and the fact that f^n is a chain map, so $\partial f^n = f^n \partial$:

$$\begin{aligned}
\partial \psi^n &= \partial f^{n+2} \circ K^n - \partial K^{n+1} \circ f^n && (9)\\
&= f^{n+2} \circ \partial \circ K^n - (-K^{n+1} \circ \partial + f^{n+2} \circ f^{n+1}) \circ f^n && (10)\\
&= f^{n+2} \circ (-K^n \circ \partial + f^{n+1} \circ f_n) - K^{n+1} \circ \partial \circ f_n + f^{n+2} \circ f^{n+1} \circ f^n && (11)\\
&= -f^{n+2} \circ K^n \circ \partial + K^{n+1} \circ \partial \circ f_n && (12)\\
&= -(f^{n+2} K^n - K^{n+1} f^n) \circ \partial && (13)\\
&= -\psi^n \circ \partial. && (14)
\end{aligned}$$

Let us show an anti-chain map ψ gives rise to a well-defined map on homology

$$\begin{aligned} \psi_* : H_i(C_*^n) &\longrightarrow H_{i+1}(C_*^{n+3}) \\ [\alpha] &\longmapsto [\psi\alpha] \end{aligned} \tag{15}$$

First, we must check it sends cycles to cycles. If α is a cycle, then $\partial\psi\alpha = -\psi\partial\alpha = 0$ as desired. To check well-definedness, suppose $\alpha + \partial\beta$ is another representative of $[\alpha]$. Then, $\psi(\alpha + \partial\beta) = \psi\alpha - \partial\psi\beta$ is homologous to $\psi\alpha$.

Solution of second part: Before proceeding, let us verify a key identity involving ψ_*^n.

Claim: On homology, we have

$$\psi_*^n f_*^{n-1} = f_*^{n+2}\psi_*^{n-1}. \tag{16}$$

Proof of Claim. We compute, for a cycle $\beta \in C_*^{n-1}$ (so $\partial\beta = 0$):

$$\begin{aligned} \psi^n f^{n-1}\beta &= (f^{n+2}K^n - K^{n+1}f^n)f^{n-1}\beta && (17) \\ &= f^{n+2}K^n f^{n-1}\beta - K^{n+1}f^n f^{n-1}\beta && (18) \\ &= f^{n+2}K^n f^{n-1}\beta - K^{n+1}(\partial K^{n-1} + K^{n-1}\partial)\beta && (19) \\ &= f^{n+2}K^n f^{n-1}\beta - K^{n+1}\partial K^{n-1}\beta && (20) \\ &= f^{n+2}K^n f^{n-1}\beta - (f^{n+2}f^{n+1} - \partial K^{n+1})K^{n-1}\beta && (21) \\ &= f^{n+2}(K^n f^{n-1} - f^{n+1}K^{n-1})\beta + \partial K^{n+1}K^{n-1}\beta && (22) \\ &= f^{n+2}\psi^{n-1}\beta + \text{(a boundary)}, && (23) \end{aligned}$$

verifying the claim.

Now, suppose that ψ^n is an isomorphism. We need to verify the sequence (8) is exact, i.e. $\ker f_*^{n+1} = \operatorname{im} f_*^n$. There are two assertions to check:

- $\operatorname{im} f_*^n \subset \ker f_*^{n+1}$: this follows immediately from the chain homotopy (5). Indeed, for a cycle $\beta \in \operatorname{im} f_*^n$, so β is homologous to $f^n\alpha$, for some cycle α, we have that

$$\begin{aligned} f^{n+1}\beta &= f^{n+1}f^n\alpha + f^{n+1}\text{(a boundary)} \\ &= (\partial K^n + K^n\partial)\alpha + f^{n+1}\text{(a boundary)} \\ &= \text{(a boundary)}, \end{aligned} \tag{24}$$

as $\partial\alpha = 0$, verifying that on homology $f_*^{n+1}[\beta] = 0$, so $\beta \in \ker f_*^{n+1}$.

- $\ker f_*^{n+1} \subset \mathrm{im}\, f_*^n$: Suppose we have a cycle $\beta \in C_*^{n+1}$ with $[\beta] \in \ker f_*^{n+1}$, so $f^{n+1}\beta = \partial\alpha$. By the isomorphism (7), β is homologous to $\psi^{n-2}\tau$, for some cycle $\tau \in C_*^{n-2}$.

  Now, using the key identity, note that

$$f^{n+1}\beta \sim f^{n+1}\psi^n\tau = \psi^{n-1}f^{n-2}\tau + (\text{ a boundary}). \tag{25}$$

  So, if $f^{n+1}\beta$ is a boundary, then $\psi^{n-1}f^{n-2}\tau$ is a boundary, which by the isomorphism (7), implies that $f^{n-2}\tau$ is a boundary, i.e.

$$f^{n-2}\tau = \partial\gamma. \tag{26}$$

  Then, note that

$$\begin{aligned}
\beta &\sim \psi^{n-2}\tau && (27)\\
&= (f^n K^{n-2} - K^{n-1}f^{n-2})\tau && (28)\\
&= f^n K^{n-2}\tau - K^{n-1}\partial\gamma && (29)\\
&= f^n K^{n-2}\tau - (f^n f^{n-1}\gamma - \partial K^{n-1}\gamma) \quad = f^n(K^{n-2}\tau - f^{n-1}\gamma) - \partial(K^{n-1}\gamma). && (30)
\end{aligned}$$

  If we can show that $\eta := (K^{n-2}\tau - f^{n-1}\gamma)$ is closed, then the above calculation will imply that $f_*^n[\eta] = [\beta]$ as desired. We check:

$$\begin{aligned}
\partial\eta &= \partial(K^{n-2}\tau - f^{n-1}\gamma) && (31)\\
&= (f^{n-1}f^{n-2}\tau - f^{n-1}\partial\gamma) && (32)\\
&= (f^{n-1}f^{n-2}\tau - f^{n-1}f^{n-2}\tau) && (33)\\
&= 0. && (34)
\end{aligned}$$

5. (14 points total) $SO(3)$ *and the quaternions.* The topological group $SO(3)$ is defined as the space of real 3 x 3 matrices A with that are *orthogonal* (meaning $A^T = A^{-1}$) and have determinant 1. The topology on $SO(3)$ is the subspace topology, coming from the inclusion of $SO(3) \subset \mathbb{R}^9$, the space of all 3 x 3 matrices.

   Let $\mathbb{H}$ denote the group of quaternions (recall that these are numbers of the form $a + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$, for $a, b, c, d \in \mathbb{R}$, with non-commutative multiplication rules determined by $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$).

   a. (6 points) A quaternion is called **pure** if it has 0 real part, i.e., it is of the form $b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$. Thinking of $\mathbb{R}^3$ as the subspace of pure quaternions in $\mathbb{H}$, any quaternion $q \in \mathbb{H}$ induces a map

   $$\begin{aligned} A_q : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto qxq^{-1}. \end{aligned} \tag{35}$$

   Show that when restricted to the unit quaternions (those with norm 1 using the usual Euclidean norm in $\mathbb{R}^4$), such a correspondence gives a (continuous) map

   $$\phi : S^3 \to SO(3). \tag{36}$$

   b. (8 points) Prove that the map ϕ is a covering map, and use it to calculate $\pi_1(SO(3))$.

Solutions: **5a.** We use the following facts: the quaternions have a multiplicative norm, defined for

$$q = a + bi + cj + dk$$

as

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

Also, the inverse of q is given by

$$\frac{a - bi - cj - dk}{\|q\|^2}$$

Since inversion of unit quaternions and multiplication of quaternions are both given by polynomials in each coordinate, they define continuous maps, so the given map

$$A_- : S^3 \to M_{3\times 3}(\mathbb{R})$$

is continuous. Since multiplication of quaternions preserves norms, the assignment

$$x \mapsto qxq^{-1}$$

preserves the norm of x as a vector in $\mathbb{R}^3$, so it defines an element of $O(3)$. Therefore A_q gives a continuous map

$$S^3 \to O(3)$$

Since S^3 is connected, this map must land in the path component of the identity, which is $SO(3)$.

5b. The map defined above

$$\phi : S^3 \to SO(3)$$

is clearly a homomorphism of groups. It is surjective because we can check that quaternions of the form $a + bi$ hit the rotations about the i-axis, and similarly $a + cj$ hits the rotations about the j-axis, and these are enough to generate the rotations of $\mathbb{R}^3$. The map has kernel $\{\pm 1\}$ since a quaternion that fixes the i axis must be of the form $a + bi$, and if it fixes the j axis it must be of the form $a + cj$, so if it fixes both then it must be real, which leaves only $+1$ and -1.

Now we will show ϕ is a covering map. It suffices to do this at the identity of $SO(3)$, since we can use left-multiplication to translate the result to all other points. We pick a small neighborhood U of the identity $1 \in SO(3)$ such that its preimage $\phi^{-1}(U)$ lies in two small balls about $+1$ and -1 in S^3. Focus on the component $V_{+1} \subset \phi^{-1}(U)$ about $+1$; we need to show that $\phi : V_{+1} \to U$ is a homeomorphism. We already know from the above that it is surjective. For injectivity, if two quaternions $q_1, q_2 \in V_{+1}$ gave the same rotation in U, then $q_1 q_2^{-1} = \pm 1$, but when U is small enough the only possibility is $+1$, so $q_1 = q_2$. Now $\phi : V_{+1} \to U$ is a continuous bijection, but by restricting to a slightly smaller closed ball $C \subset U$ and its preimage, we get a continuous bijection between a compact space and a Hausdorff one, which is therefore a homeomorphism. Restricting this to an even slightly smaller open ball $U' \subset C$, we conclude that U' is an evenly covered open set.

Now we know that ϕ is a 2-sheeted cover. Since S^3 is simply connected, it is the universal cover of $SO(3)$ and so $\pi_1(SO(3)) \cong \mathbb{Z}/2$.

6. (18 points total) *Computation via decompositions.* Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point.

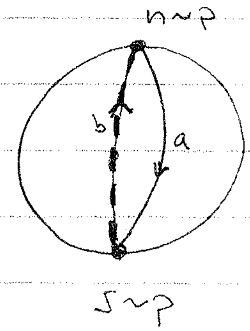
   a. (6 points) Put a cell complex structure on X and use this to compute $\pi_1(X)$.

   b. (6 points) Put a Δ-complex structure on X and use this to compute $H_*^{\Delta}(X)$, its simplicial homology for this structure.

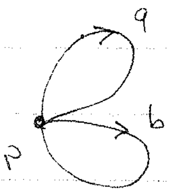
   c. (6 points) Compute the singular homology of X directly, using the Mayer-Vietoris sequence or excision.

   Solutions: See next page.

6 a.) There are many choices, but one CW structure ~~is~~ comes from cutting as follows:



i.e. $X^0 = \bullet_p$

$X^1 =$ 

X^2 consists of attaching two 2-cells to X^1, one along ba and another along $b^{-1}a^{-1}$.

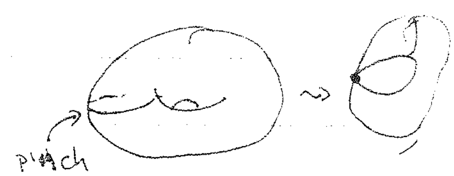
Now, $\pi_1(X^1, p) = \langle a, b \rangle$, so by Prop. 1.26 of Hatcher (which uses Van Kampen to compute the homology of CW complexes using the homotopy classes of the attaching map), we get that

$$\pi_1(X) = \pi_1(X^1) \Big/ \langle ba, b^{-1}a^{-1} \rangle$$

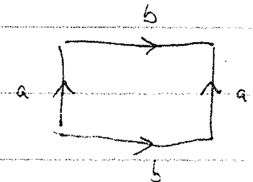
$$= \langle a, b \mid ba, b^{-1}a^{-1} \rangle = \langle ab \mid ba \rangle$$

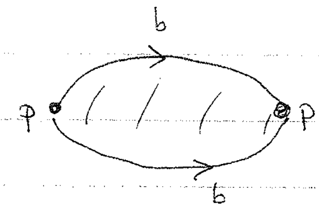
$$\cong \langle a \rangle \cong \mathbb{Z}.$$

b) Viewing X as a torus S^2 w/ a meridian circle collapsed

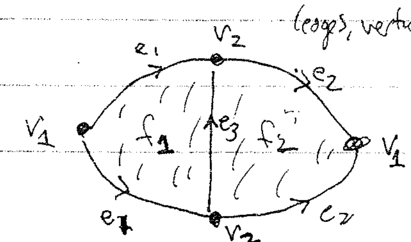


one obtains the following description:

$X =$ 

/ the circle a = 

We can put the following Δ-complex structure on this description (edges, vertices, and faces denoted by e_i's, v_i's, and f_i's.



Thus, for this Δ-complex structure, $C_i^{\Delta}(X) = \begin{cases} \mathbb{Z}v_1 \oplus \mathbb{Z}v_2, & i=0 \\ \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 & i=1 \\ \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 & i=2 \\ 0 \text{ otherwise.} \end{cases}$

The differential is:

$\partial f_1 = e_3 - e_1 + e_1 = e_3$

$\partial f_2 = e_2 - e_2 - e_3 = -e_3$

$\partial e_1 = v_2 - v_1 \qquad \partial e_3 = v_2 - v_2 = 0.$

$\partial e_2 = v_1 - v_2$

$\partial v_1 = 0, \qquad \partial v_2 = 0.$

Thus, $H_2^{\Delta}(X) = \dfrac{\ker \partial_2}{\operatorname{im} \partial_3} = \ker \partial_2 \cong \boxed{\mathbb{Z}}$, generated by $f_1 + f_2$.

$H_1^{\Delta}(X) = \dfrac{\ker \partial_1}{\operatorname{im} \partial_2} = \dfrac{\langle e_1 - e_2, e_3 \rangle}{\langle e_3 \rangle} \cong \mathbb{Z}$, generated by $[e_1 - e_2]$,

$H_0^{\Delta}(X) = \dfrac{\ker \partial_0}{\operatorname{im} \partial_1} = \dfrac{\langle v_1, v_2 \rangle}{\langle v_1 - v_2 \rangle} \cong \mathbb{Z}$, generated by, e.g. $[v_1]$;

and $H_i^{\Delta}(X) = 0$ for $i \neq 0, 1, 2$.

c) Using excision: Let $n, p \in S^2$ be the north & south poles, and $S^0 \subset S^2$ be $\{n, p\}$. Note $X = S^2/S^0$.

Claim: (S^2, S^0) is a good pair. The open nbhd deformation retracting to S^0, is a union of small balls around n, p.

Thus, we apply the LES for the reduced homology of the pair (which uses excision):

S^2/S^0

$$\cdots \to \tilde{H}_i(S^0) \longrightarrow \tilde{H}_i(S^2) \longrightarrow \tilde{H}_i(\cancel{X}) \longrightarrow \tilde{H}_{i-1}(S^0) \to \tilde{H}_{i-1}(S^2) \to \cdots$$

Let's use this to compute $\tilde{H}_i(X)$:

- If $i > 2$, then $\tilde{H}_i(S^0) \cong \tilde{H}_{i-1}(S^0)$, so from the LES:
  $0 = \tilde{H}_i(S^2) \cong \tilde{H}_i(X)$.

- If $i = 2$, then $\tilde{H}_i(S^0) = 0$, and $\tilde{H}_{i-1}(S^0) = 0$, so from the LES:
  $\mathbb{Z} = \tilde{H}_2(S^2) \cong \tilde{H}_2(X)$.

- If $i = 1$, then $\tilde{H}_i(S^2) = 0$ and $\tilde{H}_{i-1}(S^2) = 0$,
  so from the LES: $\tilde{H}_1(X) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$.

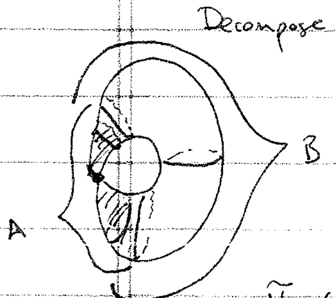
- If $i = 0$, then $\tilde{H}_i(S^2) = 0$ and $\tilde{H}_1(S^0) = 0$, so
  $\tilde{H}_i(X) = 0$.

Thus, $$\tilde{H}_i(X) = \begin{cases} \mathbb{Z} & i = 1, 2 \\ 0 & \text{otherwise}, \end{cases}$$

so $$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 1, 2 \\ 0 & \text{otherwise}. \end{cases}$$ ☒

<u>Alternately, using</u> <u>Mayer-Vietoris</u> (M-V):

Decompose X as $A \cup B$ using this picture:



We have $A \simeq pt.$, $B \simeq S^1$,
$A \cap B \simeq S^1 \amalg S^1$.

M-V gives us the following LES: (for reduced homology)

$$\cdots \to \tilde{H}_i(A \cap B) \longrightarrow \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(X) \xrightarrow{\partial_*} \tilde{H}_{i-1}(A \cap B) \to \tilde{H}_{i-1}(A) \oplus \tilde{H}_{i-1}(B) \to \cdots$$

- When $i = 0$, $\tilde{H}_0(A) \oplus \tilde{H}_0(B) \simeq 0 + 0 = 0$,
  and $\tilde{H}_{-1}(A \cap B) = 0$, so
  $\underline{\tilde{H}_0(X) = 0}$.

• when $i=1$, $\tilde{H}_1(A) = \tilde{H}_1(pt.) = 0$ and $\tilde{H}_0(A\cap B) = \mathbb{Z}$, so

(it's not zero! b/c $A\cap B$ is disconnected).

we have the exact sequence

$$\tilde{H}_1(A\cap B) \longrightarrow \tilde{H}_1(B) \xrightarrow{(i_B)_*} \tilde{H}_1(X) \longrightarrow \underset{\tilde{H}_0(A\cap B)}{\underset{\parallel}{\mathbb{Z}}} \longrightarrow 0.$$

Claim: $\tilde{H}_1(A\cap B) \longrightarrow \tilde{H}_1(B)$ is surjective.

Cor: $(i_B)_*$ is 0, so $\tilde{H}_1(X) \xrightarrow{\cong} \mathbb{Z}$. (by exactness).

Proof of claim: $A\cap B = F_1 \amalg F_2$, where each $F_i \simeq S^1$.

~~Note~~ Thus, $\tilde{H}_1(A\cap B) = \tilde{H}_1(F_1) \oplus \tilde{H}_1(F_2)$,

So it will suffice to show $\tilde{H}_1(F_1) \xrightarrow{i_*} \tilde{H}_1(B)$ is surjective,

where $i: F_1 \hookrightarrow B$ is the inclusion.

But i is a homotopy equivalence, so we're done. ☒

• When $i=2$, $\tilde{H}_2(A\cap B) = \tilde{H}_2(A) = \tilde{H}_2(B) = 0$,

$\tilde{H}_1(A\cap B) = \tilde{H}_1(F_1) \oplus \tilde{H}_1(F_2) = \mathbb{Z}\oplus\mathbb{Z}$, $\tilde{H}_1(A) = 0$,

and $\tilde{H}_1(B) = \tilde{H}_1(S^1) = \mathbb{Z}$, so we get

$$0 \longrightarrow \tilde{H}_2(X) \xrightarrow{\partial_*} \underset{\mathbb{Z}\,\oplus\,\mathbb{Z}}{\underset{\parallel}{\tilde{H}_1(F_1)\oplus\tilde{H}_1(F_2)}} \xrightarrow{j_*} \underset{\mathbb{Z}}{\underset{\parallel}{\tilde{H}_1(B)}} \to \cdots$$

Up to a choice of generators here, j_* sends (a,b) to

$\pm a \pm b$ (as $j_*|_{\tilde{H}_1(F_i)}$ is an isomorphism). And, ∂_* is injective.

Hence, $\tilde{H}_2(X) \cong$ im $\partial_* \cong \ker j_* \cong \mathbb{Z}$.

• when $i>2$, $\tilde{H}_i(A) = \tilde{H}_i(B) = \tilde{H}_{i-1}(A\cap B) = 0$, so $\tilde{H}_i(X) = 0$.

Thus, once more,

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z} & i = 1, 2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

7. (12 points total) *A covering space corresponding to a subgroup.* Let X be a wedge of three circles with basepoint p the common point at which the circles are wedged. We showed in class that the fundamental group of $\pi_1(X,p)$ is $\langle a,b,c\rangle$, the free group on three generators.

a. (7 points) Let $G \subset \pi_1(X,p)$ be the subgroup

$$G := \langle a^4, ac, c^2, ab, b^2, a^2ba^{-3}, a^2b^{-1}a^{-3}, a^2ca^{-3}, a^2c^{-1}a^{-3}\rangle. \tag{37}$$

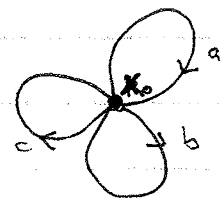
Find a covering space with basepoint

$$\pi : (\tilde{X}, \tilde{p}) \to (X, p) \tag{38}$$

corresponding to the group G.

b. (5 points) Now, using the topology of this covering space, prove that G is not a normal subgroup of $\langle a,b,c\rangle$.

Solutions: See next page.

7) a) The space X is depicted here: 

By the same ~~reasoning~~ reasoning as in §1.3 of Hatcher, to specify a covery space of X, it suffices to draw a graph, ~~and~~ ~~label~~ labeling edges a, b, or c (corresponding to the image under the covering projection), such that

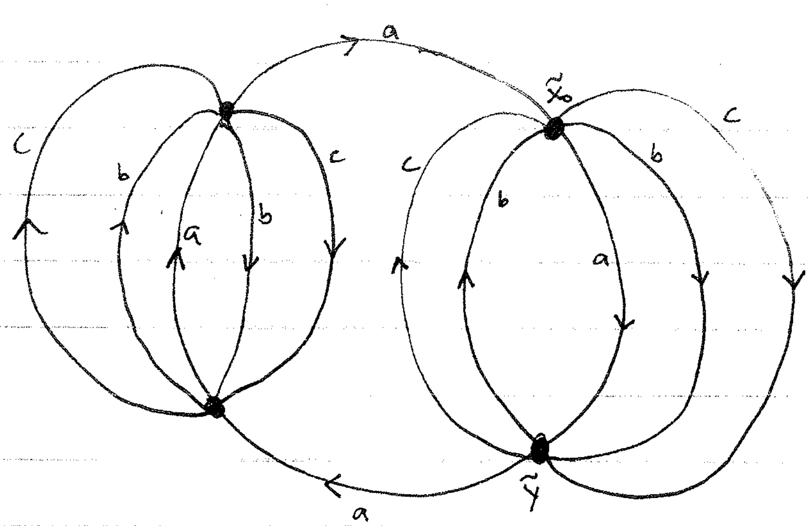
- locally, every vertex looks like a nhood of x_0 in X, i.e. there are three incoming edges labeled a, b, c, & ~~three~~ three outgoing edges labeled a, b, c.

With such a labeling, the covering morphism $p : \tilde{X} \to X$ is determined.

Now, to answer (a), we need to exhibit such a space $\tilde{X}$ with basepoint $\tilde{x}_0$, & then check that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ has the right presentation.

Various heuristics lead us to consider the following covering space:

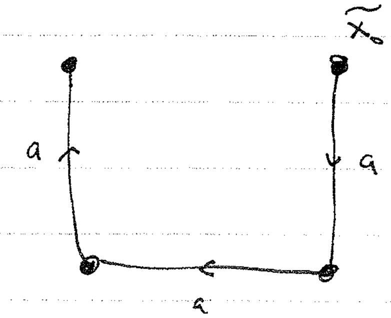
$\tilde{X}$



It is clearly a covering space of X, because it satisfies the properties described above (with edge labels determining the covering map)

To compute $\pi_1(\tilde{X},\tilde{x}_0)$, we ~~mo~~ recall that given a maximal subtree T of $\tilde{X}$, the quotient $\tilde{X} \longrightarrow \tilde{X}/T$ is a homotopy equivalence onto a wedge of circles. π_1(a wedge of ~~circles~~ S^1's) is generated by a collection of loops, each of which goes once around one of the circles, one for each S^1. Thus, $\pi_1(\tilde{X},\tilde{x}_0)$ is generated by loops projecting to such loops, i.e. loops containing exactly one edge outside of T, one for each edge outside of T.

Using this maximal subtree



,

and the above prescription,
we can thus read off generators for $p_*\pi_1(\tilde{X},\tilde{x}_0)$ (as p_* is injective):

$$\langle a^4, ab, ab^{-1}, ac, ac^{-1}, a^2ba^{-3}, a^2ca^{-3}, a^2b^{-1}a^{-3}, a^2c^{-1}a^{-3}\rangle$$

which is

$$\cong \langle a^4, ab, b^2, ac, c^2, a^2ba^{-3}, a^2ca^{-3}, a^2b^{-1}a^{-3}, a^2c^{-1}a^{-3}\rangle$$

as we've only changed those two, and one can see that

$$\langle ab, b^2, ac, c^2\rangle \cong \langle ab, ab^{-1}, ac, ac^{-1}\rangle.$$ ∎

(b) It suffices by Hatcher §1.3 to show that $\tilde{X} \xrightarrow{p} X$ is not a normal covering space, which will follow from

Claim: In the picture of $\tilde{X}$, there is no deck transformation taking $\tilde{x}_0$ to $\tilde{y}$.

Pf: A deck transformation would have to take a path labeled ba starting at $\tilde{x}_0$ to a path labeled ba starting at $\tilde{y}$. But the latter path is a closed loop, and the former is not. ∎