Math 215B Take-home Midterm

Due Tuesday February 19th, 2013 at the beginning of class (12.50 pm)

Instructions. You are welcome to use the results from the book or class that have been developed by class on Tuesday, February 12th. If you have any questions about the exam, you may e-mail or ask me in person. Discussion the problems with anyone else before the deadline will be a violation of the Stanford Honor Code. Please hand in your solutions to me in person, or slide them under my door (Building 380, room 382F). Solutions received after the deadline will not be considered.

Name:

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Stanford ID number:

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Signature acknowledging the honor code:

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1. (10 points total) *Wrong-way maps.* We have seen that singular homology is a *functorial* assignment, that is, given a map $f : X \to Y$ of topological spaces, there is an induced map $f_* : H_i(X) \to H_i(Y)$ on homology groups. In some cases, if the map $f : X \to Y$ is particularly nice, there also exists a map $f^! : H_i(Y) \to H_i(X)$, called a *wrong-way* or *transfer* map.

   a. (5 points) Let $p : \tilde{X} \to X$ be a k-sheeted covering map, for some finite k. Construct a (non-trivial!) map of chain complexes

   $$C_i(X) \longrightarrow C_i(\tilde{X}) \tag{1}$$

   and show that it is a chain map, giving rise to an induced map on homology

   $$p^! : H_i(X) \longrightarrow H_i(\tilde{X}). \tag{2}$$

   b. (5 points). Show that the composition

   $$p_* \circ p^! : H_i(X) \to H_i(X) \tag{3}$$

   is multiplication by k.

2. (14 points total) *Applications to group theory.*

   a. (7 points) Let X be a wedge sum of n circles, with its natural graph structure, and let $\tilde{X} \to X$ be a covering space with $Y \subset \tilde{X}$ a finite connected graph. Show there is a finite graph $Z \supset Y$ having the same vertices as Y, such that the projection $Y \to X$ extends to a covering space $Z \to X$.

   b. (7 points) Using the above fact if necessary, prove the following result in group theory: Let F be a finitely generated free group, $H \subset F$ a finitely generated subgroup, and $x \in F - H$. Then there is a subgroup K of finite index such that $K \supset H$ and $x \notin K$.

3. (12 points) *Spaces not distinguished by homology.* Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

4. (10 points) *Homological algebra.* Let (C^n_*, ∂) be a collection of chain complexes indexed by $n \in \mathbb{Z}$, i.e., for each $n \in \mathbb{Z}$, there is a chain complex

$$\cdots \to C^n_k \xrightarrow{\partial} C^n_{k-1} \xrightarrow{\partial} C^n_{k-2} \to \cdots . \tag{4}$$

Let $f^n_* : C^n_* \to C^{n+1}$ be a chain map, one for each n. Suppose that the composite $f_{n+1} \circ f_n : C^n \to C^{n+2}$ is chain-homotopic to zero for all n, by a chain homotopy $K^n : C^n_* \to C^{n+2}_{*+1}$; that is,

$$f^{n+1} \circ f^n = \partial K^n + K^n \partial \tag{5}$$

Show that the map

$$\psi^n := f^{n+2} \circ K^n - K^{n+1} \circ f_n \tag{6}$$

is an *anti-chain map* from $C^n_* \to C^{n+3}_*$, meaning that $\partial \circ \psi^n = -\psi^n \circ \partial$, and deduce that ψ^n gives rise to a map on homology,

$$\psi^n_* : H_i(C^n_*, \partial) \longrightarrow H_{i+1}(C^{n+3}_*, \partial) \tag{7}$$

for all n and i. Finally, suppose that (7) is an isomorphism for all n and i. Deduce that the sequence

$$\cdots \longrightarrow H_i(C^n_*, \partial) \xrightarrow{f^n_*} H_i(C^{n+1}_*, \partial) \xrightarrow{f^{n+1}_*} H_i(C^{n+2}, \partial) \longrightarrow \cdots \tag{8}$$

is exact.

5. (14 points total) $SO(3)$ *and the quaternions.* The topological group $SO(3)$ is defined as the space of real 3 x 3 matrices A with that are *orthogonal* (meaning $A^T = A^{-1}$) and have determinant 1. The topology on $SO(3)$ is the subspace topology, coming from the inclusion of $SO(3) \subset \mathbb{R}^9$, the space of all 3 x 3 matrices.

   Let $\mathbb{H}$ denote the group of quaternions (recall that these are numbers of the form $a + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$, for $a, b, c, d \in \mathbb{R}$, with non-commutative multiplication rules determined by $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$).

   a. (6 points) A quaternion is called **pure** if it has 0 real part, i.e., it is of the form $b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$. Thinking of $\mathbb{R}^3$ as the subspace of pure quaternions in $\mathbb{H}$, any quaternion $q \in \mathbb{H}$ induces a map

$$\begin{aligned} A_q : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto qxq^{-1}. \end{aligned} \tag{9}$$

Show that when restricted to the unit quaternions (those with norm 1 using the usual Euclidean norm in $\mathbb{R}^4$), such a correspondence gives a (continuous) map

$$\phi : S^3 \to SO(3). \tag{10}$$

b. (8 points) Prove that the map ϕ is a covering map, and use it to calculate $\pi_1(SO(3))$.

6. (18 points total) *Computation via decompositions.* Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point.

   a. (6 points) Put a cell complex structure on X and use this to compute $\pi_1(X)$.

   b. (6 points) Put a Δ-complex structure on X and use this to compute $H_*^{\Delta}(X)$, its simplicial homology for this structure.

   c. (6 points) Compute the singular homology of X directly, using the Mayer-Vietoris sequence or excision.

7. (12 points total) *A covering space corresponding to a subgroup.* Let X be a wedge of three circles with basepoint p the common point at which the circles are wedged. We showed in class that the fundamental group of $\pi_1(X, p)$ is $\langle a, b, c\rangle$, the free group on three generators.

   a. (7 points) Let $G \subset \pi_1(X, p)$ be the subgroup

   $$G := \langle a^4, ac, c^2, ab, b^2, a^2ba^{-3}, a^2b^{-1}a^{-3}, a^2ca^{-3}, a^2c^{-1}a^{-3}\rangle. \tag{11}$$

   Find a covering space with basepoint

   $$\pi : (\tilde{X}, \tilde{p}) \to (X, p) \tag{12}$$

   corresponding to the group G.

   b. (5 points) Now, using the topology of this covering space, prove that G is not a normal subgroup of $\langle a, b, c\rangle$.